ENS de Lyon 2017 - 2018

MASTER 1–ALGEBRAIC GEOMETRY WEEK 2, 22.01.2018

TD 2-Spectrum of a ring I-topological aspects

If not explicitly mentioned, A is always an arbitrary ring, $X = \mathrm{Spec}(A)$ is endowed with the Zariski topology and for $f \in A$ and I an ideal of A we let

$$V(I) = \{\mathfrak{p} \in X |\, I \subset \mathfrak{p}\}, \quad D(f) = X \setminus V((f)) = \{\mathfrak{p} \in X |\, f \notin \mathfrak{p}\}.$$

If $\mathfrak{p} \in X$ let $k(\mathfrak{p}) = \mathrm{Frac}(A/\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ be its residue field. Finally, if $f : A \to B$ is a map of rings, the associated map on Spec is the (continuous) map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ sending $\mathfrak{p}$ to $f^{-1}(\mathfrak{p})$. If S is a subset of a topological space, we write $\overline{S}$ for the closure of S in that space. Finally, k is always a field.

0.1 Basic things

1. Describe the spectrum of $\mathbf{Z}$, of a discrete valuation ring and of $\mathbf{Z}[T]/(T^2+1)$, $\mathbf{Z}[T]/(T^n)$ for $n \geq 1$.
2. For $\mathfrak{p} \in X$, what is the closure of $\{\mathfrak{p}\}$ in X ? When is $\{\mathfrak{p}\}$ closed in X ? If $\mathfrak{q} \in X$, what does it mean concretely that $\mathfrak{q} \in \overline{\{\mathfrak{p}\}}$ (we say that $\mathfrak{p}$ **specializes** to $\mathfrak{q}$) ? What are the closed points of $\mathrm{Spec}(k[X,Y])$?
3. Given a multiplicative subset S of A and an ideal I of A, identify (in a way compatible with topologies !) $\mathrm{Spec}(S^{-1}A)$ and $\mathrm{Spec}(A/I)$ with subspaces of $\mathrm{Spec}(A)$. Is the image of $\mathrm{Spec}(S^{-1}A)$ always open ?
4. Prove that if $(f_i)_{i \in I}, g$ are elements of A, then $D(g) \subset \cup_{i \in I} D(f_i)$ if and only if $g \in \sqrt{(f_i)_{i \in I}}$. Prove that the open subsets of X that are quasi-compact [1] are exactly the subsets $X \setminus V(I)$ for finitely generated ideals I of A, or equivalently the finite unions of $D(f)$'s.

0.2 Irreducibility

A nonempty topological space X is **irreducible** if it cannot be written as the union of two closed subsets different from X. A subset of X is called irreducible it it's an irreducible space for the induced topology.

1. Prove that X is irreducible if and only if each nonempty open subset of X is dense in X. Also, prove that a subset S of X is irreducible iff its closure in X is so, and the image of an irreducible subset of X by a continuous map $f : X \to Y$ is irreducible.
2. Prove that $\mathrm{Spec}(A)$ is irreducible iff $A/\mathrm{nil}(A)$ is an integral domain, and construct a natural bijection between irreducible closed subsets of $\mathrm{Spec}(A)$ and prime ideals of A.

0.3 Fundamental tools

Keep the following properties in mind, you will use them all the time ! Let $f : A \to B$ be a morphism of rings and let $\varphi : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be the induced morphism.

1. Prove that φ has dense image if and only if $\ker(f)$ consists of nilpotent elements.
2. Prove that if $\mathfrak{p} \in \mathrm{Spec}(A)$, then $\varphi^{-1}(\mathfrak{p})$ with the induced topology is homeomorphic to $\mathrm{Spec}(k(\mathfrak{p}) \otimes_A B)$. **Hint** : $k(\mathfrak{p}) \otimes_A B$ is the same as $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$.
3. Prove that if φ is surjective, then $\mathrm{Spec}(A' \otimes_A B) \to \mathrm{Spec}(A')$ is surjective for all morphisms $A \to A'$. **Hint** : use the previous result.
4. Prove that if f is finite, then φ has finite fibers. What about the converse ?
5. Describe $\mathrm{Spec}(\mathbf{Z}[T])$ and $\mathrm{Spec}(k[X,Y])$. **Hint** : consider the fibres of $\mathrm{Spec}(\mathbf{Z}[T]) \to \mathrm{Spec}(\mathbf{Z})$.

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1. A topological space X is quasi-compact if any open covering of X can be refined to a finite sub-covering.

0.4 Integrality : Going up

Let $f : A \to B$ be an integral map of rings, and let $\varphi : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be the induced map.

1. Prove that if f is injective, then φ is surjective. **Hint** : reduce to the case when A is local.
2. Prove that φ is a closed map, and more precisely for all ideals I of B we have $\varphi(V(I)) = V(f^{-1}(I))$.
3. Prove the **going up theorem** : $\varphi : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is specializing, i.e. if $\mathfrak{p} \subset \mathfrak{p}'$ are prime ideals of A and $\mathfrak{q}$ is a prime of B over $\mathfrak{p}$, then we can find a prime $\mathfrak{q} \subset \mathfrak{q}'$ of B over $\mathfrak{p}'$.
4. Prove that the map on spectra induced by $k[X,Y]/(X^2 - Y^3) \to k[t]$, $X \to t^3, Y \to t^2$ is a homeomorphism. Is this map of rings an isomorphism ?

0.5 Finitely generated algebras over a field

Let k be a field and let A be a finitely generated k algebra. Set $X = \mathrm{Spec}(A)$.

1. Prove that any nonempty locally closed subset of X meets the set X_0 of closed points of X, and give a natural bijection between the open subsets of X_0 (with induced topology) and those of X.
2. Prove that X is a finite set iff A is finite dimensional over k, iff X is discrete.

0.6 Clopen subsets of a spectrum

We write clopen instead of "closed and open". Let $X = \mathrm{Spec}(A)$.

1. Prove that if e is an idempotent of A, then $V(eA)$ is a clopen subset of X, and if e, f are distinct idempotents of A then $V(eA) \neq V(fA)$.
2. (slightly tricky [2], but very important) Prove that $e \to V(eA)$ is a bijection between idempotents of A and clopens of X. Deduce that X is connected if and only if $0, 1$ are the only idempotents of A, or equivalently A is not the product of two nonzero rings.
3. (more difficult) Let Id be the collection of ideals of A generated by idempotents of A. Prove that $I \to V(I)$ is a bijection between Id and the subsets of X that are intersections of clopens of X. Also, the connected component of $x \in X$ is the intersection of all clopens of X containing x.

0.7 Products of varieties

1. Let B, C be A-algebras. Prove that $\mathrm{Spec}(B/nil(B) \otimes_A C) \to \mathrm{Spec}(B \otimes_A C)$ is a homeomorphism.
2. (important) Let k be an algebraically closed field and let A, B be finitely generated k-algebras.
   a) Prove that if A, B are reduced (resp. integral domains), then so is $A \otimes_k B$.
   b) Prove that if $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$ are irreducible (resp. connected), then so is $\mathrm{Spec}(A \otimes_k B)$. Give counter-examples when k is no longer algebraically closed.
3. (more difficult) Let A, B be k-algebras (not necessarily finitely generated), with k a separably closed field. Prove that if $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$ are both connected (resp. both irreducible), then so is $\mathrm{Spec}(A \otimes_k B)$.

0.8 (faithful) Flatness/integrality : Going down

1. Let $A \to B$ be a flat map of rings. Prove that the following statements are equivalent (we then say that the map is **faithfully flat**) and are satisfied for a flat local map of local rings :
   a) For any nonzero A-module M we have $M \otimes_A B \neq 0$.
   b) $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective.
   c) The image of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ contains all closed points of $\mathrm{Spec}(A)$
   A map of rings $A \to B$ has the **going down property** if the map on Spec is generalizing, i.e. if whenever $\mathfrak{p} \subset \mathfrak{p}'$ are prime ideals of A and $\mathfrak{q}'$ is a prime over $\mathfrak{p}'$, there is a prime $\mathfrak{q} \subset \mathfrak{q}'$ over $\mathfrak{p}$.
2. Prove that any flat map of rings has the going down property. **Hint** : use the previous item.
3. (difficult) Let $A \to B$ be an integral injection of integral domains, with A normal (i.e. integrally closed in its field of fractions). Prove that $A \to B$ has the going down property, as follows : first reduce to proving that $ab \notin \mathfrak{p}B$ when $b \notin \mathfrak{q}'$ and $a \notin \mathfrak{p}$, then show that any element of $\mathfrak{p}B$ satisfies an equation of the form $x^n + a_{n-1}x^{n-1} + ... + a_0 = 0$ with $a_i \in \mathfrak{p}$, and finally study the minimal polynomial of b over $\mathrm{Frac}(A)$.

2. Since we are lacking sheaves... they will come...

0.9 Quotients by finite groups

Let A be a ring and let G be a finite group acting by automorphisms of rings on A. Let A^G be the subring of invariant elements and let $\pi : \mathrm{Spec}(A) \to \mathrm{Spec}(A^G)$ be the natural map.

1. Prove that the natural morphism $A^G \to A$ is integral. Deduce that π is surjective.
2. Prove that the fibers of π are precisely the G-orbits for the natural action of G on $\mathrm{Spec}(A)$. **Hint** : use the map $a \to \prod_{g\in G} g(a)$ and the prime avoidance lemma.
3. Prove that π is open. **Hint** : if $a \in A$, prove that $\pi(D(a)) = \cup_i D(b_i)$, where

$$\prod_{g\in G}(T - g.a) = T^d + b_{d-1}T^{d-1} + ... + b_0.$$

0.10 Chevalley's constructibility theorem, universally open maps

Let A be a ring. A subset C of $X = \mathrm{Spec}(A)$ is called **constructible** if it is a finite union of spaces of the form $U \cap (X \setminus V)$, with U, V quasi-compact open subsets of X.

1. Prove that $C \subset X$ is constructible if and only if it is a finite union of sets of the form $D(f) \cap V(I)$, with $f \in A$ and I a finitely generated ideal.
2. Prove that the collection of constructible subsets is stable under finite unions, complement and finite intersections.
3. Prove that if $f \in A$ and I is a finitely generated ideal of A, then the maps $\mathrm{Spec}(A_f) \to \mathrm{Spec}(A)$ and $\mathrm{Spec}(A/I) \to \mathrm{Spec}(A)$ send constructible subsets to constructible subsets.
4. (hard) We want to prove **Chevalley's constructibility theorem** : if B is a **finitely presented** A-algebra, i.e. $B = A[X_1, ..., X_n]/(f_1, ..., f_m)$ for some $f_i \in A[X_1, ..., X_n]$ and some m, n, then $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ sends constructible subsets to constructible subsets.

   a) Show that we may assume that $n = 1$ and $B = A[X_1]$. We write $X = X_1$ from now on and consider the natural map $\varphi : \mathrm{Spec}(A[X]) \to \mathrm{Spec}(A)$.

   b) Prove that φ is open, more precisely if $f = a_0 + a_1X + ... + a_nX^n$, then $\varphi(D(f)) = \cup_{i=0}^n D(a_i)$.

   c) Let $f, g \in A[X]$, with g monic. Prove that there are $a_1, ..., a_n \in A$ such that

   $$\varphi(D(f) \cap V(g)) = D(a_1) \cup ... \cup D(a_n).$$

   Hint : consider the characteristic polynomial of multiplication by f on $A[X]/(g)$.

   d) Prove Chevalley's theorem. **Hint** : to prove that the image of $D(f) \cap V(g_1, ..., g_k)$ is constructible (for $f, g_i \in A[X]$) make a double induction on k and (inside the first) on $\sum_i \deg(g_i)$.
5. Prove that for any morphism of rings $A \to B$, the inverse image of a constructible subset of $\mathrm{Spec}(A)$ in $\mathrm{Spec}(B)$ is constructible. Also, prove that any constructible subset of $\mathrm{Spec}(A)$ is the image of a map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ for some finitely presented A-algebra $A \to B$. **Hint** : this is much easier and doesn't use Chevalley's theorem.
6. a) Let C be a constructible subset of $X = \mathrm{Spec}(A)$. Prove that any element x of the closure of C in X is the specialization of a point of C, i.e. there is $y \in C$ such that $x \in \overline{\{y\}}$.

   b) (important result) Prove that if B is a flat and finitely presented A-algebra (A being any ring), then the map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is open. **Hint** : use Chevalley, part a) and the going-down property.

   c) (very nice) Prove that if k is a field, then for any k-algebras A, B the map $\mathrm{Spec}(A \otimes_k B) \to \mathrm{Spec}(B)$ is open.